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Planck distribution for a complex q-boson gas

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Abstract. The q-deformed boson gas at finite temperature is investigated for the quantum deformation parameter q to be complex ($s=\ln q=a+ib$). It is shown that the real part of the energy density distribution corresponds to the Planck energy density distribution for a suitable range of the parameter b, which restricts the maximum number of quantum states in the model of the system. The spectra of q-oscillator are different for different (a, b)-values. The classical dimensionless constant, characterizing the Wien shift law, is presented as a function of (a, b) and, in the spirit of the theory of q-deformed boson gas, it is possible to interpret it as the radiation of a non-ideal black-body or a real body. The role of the imaginary part of oscillator energy is also studied.

Martin-Delgado [1] has recently studied the q-deformation of the Planck energy density distribution of a boson gas (the black-body radiation), with the bosons taken to obey the quantum deformed algebra of the harmonic oscillator (the $SU_q(2)$) [2]. For $x(=\hbar\omega/kT)\gg 1$ i.e. for higher frequencies and/or low temperatures, he obtained a q-deformed Wien law, adding only the first two terms of the infinite sum. However, the same law is also used for $x\in[0, \infty]$, which resulted in showing the role of q-deformation parameter $s(=\ln q)$ resembling that of temperature T in the energy density distribution. The maximum $x_{\max}$ of the q-deformed Wien energy distribution, which gives the q-deformed Wien shift law, is interpreted in terms of the modification of the Planck constant $\hbar$ due to the q-deformation parameter s. Such an interpretation comes from the opposing roles of s and T in $x_{\max}$(or $\omega_{\max}$), with s taken to resemble T. This is one possible interpretation of the parameter q when s has a real value. For this special case of real s, the energy eigenvalue of the q-oscillator is positive, nonlinearly increasing function of quantum number n, the number of energy states, and the sum in the partition function quickly converges compared with the case of the normal ($q\to 1$ or $s\to 0$) oscillator (see figure 1, curves 1 and 4).

In view of the earlier works [3–6] on quantum groups, the parameter $q(=e^s)$ can be complex ($s=a+ib$). This generalization allows us to consider many kinds of energy spectra of the q-oscillator, which are interesting from the theoretical point of view and need experimental verification.

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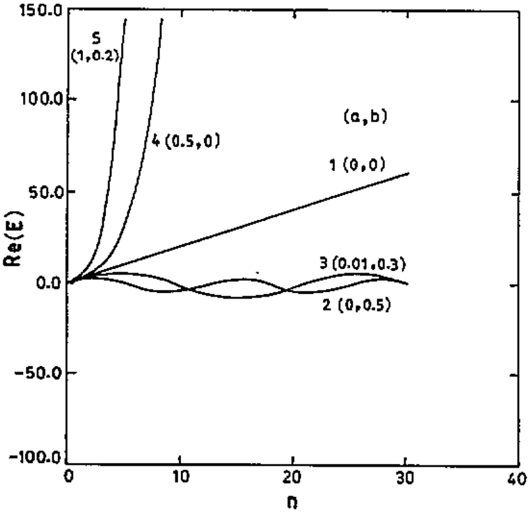

Figure 1. Real part of the energy Re(E) (in units of $\hbar\omega/2$) as a function of the quantum number n, the number of energy states. The parameters (a, b) in $s=a+ib$ have the following values: $a=0, b=0$ (curve 1); 0, 0.5 (curve 2); 0.01, 0.3 (curve 3); 0.5, 0 (curve 4); 1, 0.2 curve (5). For the curve 5, the maximum value of quantum number n is 5, i.e. $n_{max}=5$.

The energy eigenvalues $E(q, n)$ for a q-harmonic oscillator, with zero point energy subtracted, are [2]

$$E(q, n)=\tfrac{1}{2}\hbar\omega([n]+[n+1]-1). \tag{1}$$

The square bracket in (1) introduces the parameter q (or s) of the $SU_q(2)$ algebra by defining [2]

$$[x]=\frac{q^x-q^{-x}}{q-q^{-1}}=\frac{e^{sx}-e^{-sx}}{e^s-e^{-s}}=\frac{\sinh sx}{\sinh s} \tag{2}$$

where, $q=e^s$, $s=a+ib$ and a, b are real numbers. Notice the symmetry of (2) for $q\to q^{-1}$. For $q\to 1$ (or $s\to 0$) we obtain $[x]\to x$ and hence $E(q, n)\to\hbar\omega n$, the energy of the simple classical oscillator. When q is complex the energy eigenvalue also becomes complex:

$$E(q, n)=\mathrm{Re}\,E(q, n)+\mathrm{i}\,\mathrm{Im}\,E(q, n) \tag{3a}$$

where

$$\mathrm{Re}\,E(q, n)=\frac{\hbar\omega}{2}\lambda_{1,n} \tag{3b}$$

$$\mathrm{Im}\,E(q, n)=\frac{\hbar\omega}{2}\lambda_{2,n} \tag{3c}$$

with $\lambda_{1,n}$ and $\lambda_{2,n}$ as the real functions of n, a and b:

$$\lambda_{1,n}=\frac{\cosh(n+1)a\cos nb-\cosh na\cos(n+1)b}{\cosh a-\cosh b}-1 \tag{4a}$$

$$\lambda_{2,n}=\frac{\sinh(n+1)a\sin nb-\sinh na\sin(n+1)b}{\cosh a-\cosh b}. \tag{4b}$$

For a boson gas in canonical ensemble, obeying the q-deformed harmonic oscillator algebra [2], the partition function $Z(q, \omega, T)$ is defined [1] as

$$Z(q, \omega, T)=\sum_{n=0}^{n_{\max}} \mathrm{e}^{-\beta E(q,n)} \qquad \left(\beta=\frac{1}{kT}\right) \tag{5a}$$

or, equivalently, for complex q

$$\begin{aligned} Z(q, \omega, T)=&\sum_{n=0}^{n_{\max}} \mathrm{e}^{(-\beta\hbar\omega/2)\lambda_{1,n}} \cos\left(\frac{\beta\hbar\omega}{2}\lambda_{2,n}\right) \\ &+\mathrm{i}\sum_{n=0}^{n_{\max}} \mathrm{e}^{(-\beta\hbar\omega/2)\lambda_{1,n}} \sin\left(\frac{\beta\hbar\omega}{2}\lambda_{2,n}\right). \end{aligned} \tag{5b}$$

Here, we have written the upper limit as $n_{\max}$ and not infinity since the numerical calculations are always carried out with finite $n(=n_{\max})$. We enumerate the energy levels in an order that the real part of energy is a positive increasing function of quantum number n, varied from $n=0$ to $n_{\max}$. From the physical point of view only these states are interesting because the probability of occupation of a higher energy level (proportional to $|\exp(-\beta E(q, n)|)$ at finite temperature should be smaller than the lower one. We can choose $n_{\max}$ from the convergence condition of the series (5*b*). These series converge absolutely if the ratio of the $(n+1)$th and nth terms is smaller than 1. From (4*a*), (4*b*) and (5*b*) one needs

$$\exp\left\{-\frac{x}{2}(\lambda_{1,n+1}-\lambda_{1,n})\right\}<1$$

or, equally,

$$\exp\{-x\,\mathrm{e}^{(n+1)a}\cos(n+1)b\}<1 \tag{6a}$$

with

$$x=\frac{\hbar\omega}{kT}. \tag{6b}$$

From (6*a*) one obtains

$$n<\frac{\pi}{2b}-1.$$

Then, $n_{\max}$ may be defined as

$$n_{\max}=\left\{\frac{\pi}{2b}\right\}-2 \tag{7a}$$

where the brackets $\{-\}$ mean the integral part of $\pi/2b$. According to (7*a*) the physically interesting values of b, when $n_{\max}$ varies from 0 to ∞, are included in the following interval

$$0\leqslant b\leqslant\pi/4. \tag{7b}$$

For the classical and Martin-Delgado [1] cases, $b=0$; then $n_{\max}=\infty$. Practically, for the q-deformed boson case when s is real and $s(=a)>1$ only the first few terms in the sum of the partition function are needed (see (6*a*)). It is clear from (6*a*) that the two-terms approximation used in [1] is suitable for the case of large a and large x.

The energy density for the q-boson gas is written as [1]

$$U(q, \omega, T)=\frac{8\pi}{C^3}\left(\frac{\omega}{2\pi}\right)^2\left(-\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right). \tag{8}$$

Then, for complex q, one obtains from (3*a*) to (5*b*) the following equations for the real and imaginary parts of U:

$$\operatorname{Re} U(q, x)=\tfrac{1}{2}x^3\frac{X_1X_2-Y_1Y_2}{X_2^2+Y_2^2} \tag{9a}$$

$$\operatorname{Im} U(q, x)=\tfrac{1}{2}x^3\frac{X_1Y_2+X_2Y_1}{X_2^2+Y_2^2} \tag{9b}$$

where

$$X_1=\sum_{n=0}^{n_{\max}}\exp(-\tfrac{1}{2}x\lambda_{1,n})\{\lambda_{1,n}\cos(\tfrac{1}{2}x\lambda_{2,n})+\lambda_{2,n}\sin(\tfrac{1}{2}x\lambda_{2,n})\} \tag{10a}$$

$$X_2=\sum_{n=0}^{n_{\max}}\exp(-\tfrac{1}{2}x\lambda_{1,n})\cos(\tfrac{1}{2}x\lambda_{2,n}) \tag{10b}$$

$$Y_1=\sum_{n=0}^{n_{\max}}\exp(-\tfrac{1}{2}x\lambda_{1,n})\{\lambda_{2,n}\cos(\tfrac{1}{2}x\lambda_{2,n})-\lambda_{1,n}\sin(\tfrac{1}{2}x\lambda_{2,n})\} \tag{10c}$$

$$Y_2=\sum_{n=0}^{n_{\max}}\exp(-\tfrac{1}{2}x\lambda_{1,n})\sin(\tfrac{1}{2}x\lambda_{2,n}). \tag{10d}$$

Now, we investigate the Wien regime ($x\gg 1$, large energy and/or low temperature) for the system with a limited number of energy levels, defined by parameter b via (7). The two-terms approximation gives the following complex-q extension of Wien law for the real part of the energy density:

$$\begin{aligned}\operatorname{Re} U(q, x)\approx x^3\exp(-x\cosh a\cos b)\{&\cosh a\cos b\cos(x\sinh a\sin b)\\ &+\sinh a\sin b\sin(x\sinh a\sin b)\}.\end{aligned} \tag{11}$$

For $b=0$ we obtain the Martin-Delgado result [1] and the limit $q\to 1$ ($a\to 0, b\to 0$) reproduces the classical Wien law. The maximum $\operatorname{Re} U$ occurs at x_m which is the solution of the following trancendental equation:

$$\begin{aligned}&\sinh a\sin b\tan(x_m\sinh a\sin b)\\ &\quad=\frac{x_m(\cosh^2 a\cos^2 b-\sinh^2 a\sin^2 b)-3\cosh a\cos b}{3-2x_m\cosh a\cos b}.\end{aligned} \tag{12}$$

The solution x_m can be written as follows

$$x_m=f(a, b) \tag{13a}$$

where, in special cases, the function f has the following forms:

$$f(a, 0)=3/\cosh a \tag{13b}$$

$$f(0, b)=3/\cos b. \tag{13c}$$

Notice that (13*b*) is the result of Martin-Delgado [1], which is generalized here to the case of complex q.

The complex-q Stefan law is obtained in the two-terms approximation by integrating (11):

$$U^{\mathrm{stef}}(q,T)=\frac{12(k_{\mathrm{B}}T)^4}{\pi c^3\hbar^3(\cosh a\cos b)^3}\,\frac{\{1-2(\tanh a\tan b)^2-3(\tanh a\tan b)^4\}}{\{1+(\tanh a\tan b)^2\}^4}. \tag{14}$$

We note from (14) that, for a and b related as follows, the q-oscillator is non-radiating:

$$\tanh a\tan b=\pm 1/\sqrt{3}. \tag{15}$$

In the two-terms approximation, the maximum b-value is $b=\pi/8$ (see (7*a*)). Then (15) has no solution for a. In general, however, the Stefan–Boltzmann constant is reduced in comparison with the classical case.

Similarly, for $x \ll 1$ (the Rayleigh–Jeans regime) it is of interest to find the corresponding value of $n_{\max}$. However, in this case, the x^3 multiplier in (11) dominates and the $n_{\max}$-value used above can be used here too.

Figure 1 shows the Re $E(q,n)$ (in units of $\hbar\omega/2$) as a function of the quantum number n. For the classical case of $q=1$ ($a=0, b=0$), this dependence is linear (curve 1). The curves 2 and 3, corresponding to ($a=0, b=0.5$) and ($a=0.01, b=0.3$) respectively, result in oscillating spectra. The curve 4 spectrum for real s ($b=0$) is a monotonically increasing function of n (here real $s=a=0.5$). The curve 5 for ($a=1, b=0.2$) is the case of a limited number of the energy levels, where the real part of energy is an almost exponentially increasing function of n for $n\leqslant 5$ i.e. $n_{\max}=5$. For $n>5$ the energy changes from a large positive value to a negative one and loses physical meaning. Thus, different (a, b)-values give rise to different energy spectra and hence different $n_{\max}$-values for convergence of the series (5*b*).

Figure 2 illustrates the real part of the energy density Re $U(q,x)$ for several values of deformation parameter s. We notice that the Planck distribution is obtained not

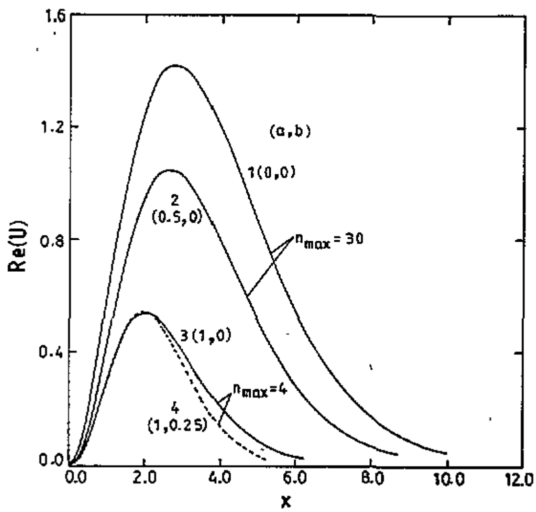

Figure 2. Real part of the energy density Re(U) as a function of the dimensionless variable $x(=\hbar\omega/kT)$. The values of the parameters (a, b) and number of terms included in the calculations have the following values: $a=0$, $b=0$, $n_{\max}=30$ (curve 1); 0.5, 0, 30 (curve 2); 1, 0, 4 (curve 3); 1, 0.25, 4 (curve 4).

only for $b=0$ (curves 1, 2 and 3) but also for $b \neq 0$ (curve 4). When q is real $(a \neq 0, b=0)$ the role of a is similar to that of the temperature on the energy density distribution, as already pointed out by Martin-Delgado [1]. The same is not true when only the imaginary term b is varied (see curves 3 and 4). Comparing curves 3 and 4, which have the same a-value but different b, one sees that Re $U(x)$ for $x>2$ is strongly depressed for the $b \neq 0$ case. In each case, however, the low frequency and/or high temperature (Rayleigh–Jeans) region of $x \ll 1$ is not affected much because of the x^3 multiplier in (11). The curves 1 and 2 are plotted for summing the $n_{\max}=30$ terms and the curves 3 and 4 are for $n_{\max}=4$ terms, in accordance with (7*a*).

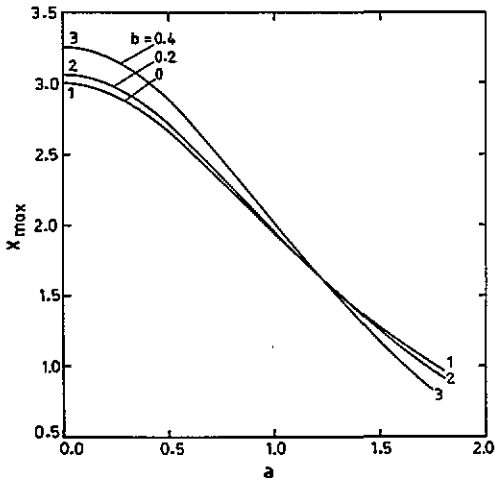

Figure 3. The Wien shift law in a q-deformed boson gas model for radiation of a black-body. The $x_{\max}=f(a, b)$ (see text, equation (13*a*)) as a function of deformation parameter (a, b). $b=0$ (curve 1); 0.2 (curve 2); 0.4 (curve 3).

Figure 3 shows how the universal constant $f(a, b)$ or $x_{\max}$ (equal to 3 for the classical case of $a=0, b=0$) vary with deformation parameter q. The curves 2 and 3 show that for $a \leqslant 1.2$ and $b \neq 0$ the maximum of the energy density distribution of a radiating body at fixed temperature (curve 1) is shifted to smaller wavelength ($x_{\max}$ may be expressed as $(hc/k)/\lambda_{\max}T$). This result may be relevant for the radiation of a non-ideal black-body (a real body) and is being subjected to experimental verification at Hanoi [7]).

Figure 4 illustrates the dependence of the imaginary part of the energy Im(E) (in units of $\hbar\omega/2$) on the a, b parameters. For $b \neq 0$, Im(E) is increased suddenly with increasing quantum number n—almost exponentially for large a. Since the imaginary part of the energy means the probability of damping of the quantum state, it is convincing that in the case of the complex q-deformation parameter the first few terms approximation in the series for the partition function is reasonable when q is chosen within a suitable range. For the same reason the model system with limited quantum states, taken numerically in increasing order of energy and quantum number, has been investigated here.

Summarizing, we have shown that for the complex q-deformation parameter the q-boson gas results in many varied energy spectra. This ranges from a monotonically increasing function for Re E as a function of the number of energy states to an oscillating one. Also, the case of a limited number of energy states is presented, which fixes the

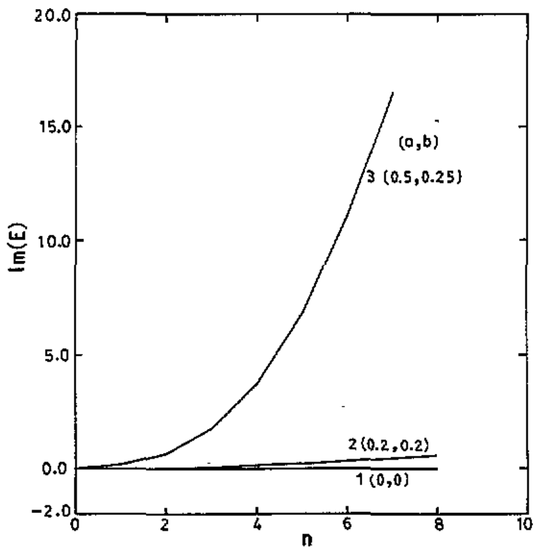

Figure 4. Imaginary part of the energy $\mathrm{Im}(E)$ (in units of $\hbar\omega/2$) as a function of the quantum number n, the number of energy states. The parameters (a, b) have the following values: $a=0$, $b=0$ (curve 1); 0.2, 0.2 (curve 2); 0.5, 0.25 (curve 3).

value of imaginary term b in the complex q-deformation parameter s $(=a+\mathrm{i}b)$. For the fixed range of b-values, the Planck distribution is obtained not only for real s $(=a)$ but also for complex s. However, it is only the real parameter a that behaves like the temperature for the energy density distribution.

Both the q-deformed Wien shift law and the Stefan–Boltzmann law are obtained for the complex s. The resulting q-Wien shift law is interpreted to be the case of a non-ideal black-body (real body) radiation, which needs experimental verification.

Finally, the Im E as a function of the number of energy states is studied for different (a, b) parameters. Its role as the damping probability of quantum states is illuminated.

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